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Two-fold photoelectron counting statistics: the clipped correlation function

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Abstract. Two-fold photoelectron counting statistics are studied for quasimonochromatic light of arbitrary spectral profile. A numerical scheme is developed for the evaluation of the double generating function from which the joint probabilities, the photoelectron correlation function, and the clipped photoelectron correlation function can be evaluated. The time need not be short compared to the coherence time as in previous studies. Numerical results are presented for the photoelectron correlation function and for the clipped correlation function for the spectrum which would be observed in a Brillouin scattering experiment.

1. Introduction

There has been considerable interest recently in determining optical spectra by examining the statistics governing the emission of photoelectrons as evinced by the review articles of Glauber (1969), Arecchi (1969), Pike (1970), and Mehta (1970). The simplest description of these statistics is the single-fold (or first-order) probability that n photoelectrons will be counted during a specified counting time. Unfortunately, these first-order probabilities are relatively insensitive to the detailed shape of the spectrum (Barakat and Glauber 1973) which limits their usefulness as an experimental tool. A more useful characterization of the arrival statistics is contained in the two-fold (second-order) probability $P(n, t_1; m, t_2)$ which is the probability that n photoelectrons will be counted during t_1 and subsequently m counted during t_2. A disadvantage of the direct use of $P(n, t_1; m, t_2)$ is the cumbersome nature of this description. A more compact second-order description is the photoelectron correlation function, of which a variant is generally measured in photoelectron counting experiments.

The purpose of this paper is to present a procedure for evaluating the double generating function, and from this, the joint probabilities and correlation function for photoelectron emissions. The method is not limited to short counting times; indeed one important result of these calculations is to display the effects of using counting times of various lengths in performing a PCS experiment. As an example we apply our general solution to a spectrum consisting of the superposition of three Lorentz lines, as would be observed in an experiment to observe Brillouin scattering. We emphasize, however, that the techniques themselves are directly applicable to gaussian light with any spectrum for which the field correlation function can be evaluated. We have examined other spectra of physical interest and these results will be published separately.

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We should like to point out that Jakeman has already examined the two-fold counting statistics in the specific case of a single Lorentz line, in that he has evaluated the generating function in closed form (Jakeman 1970). His method depends critically on the fact that the spectrum is Lorentz. Reference is also made to a recent paper of Srinivasan and Sukavanam (1972) which attempts a generalization to arbitrary spectra of Jakeman's approach. Finally, we note that it is the clipped correlation function and not the correlation function of photoelectrons itself which is commonly measured in such photoelectron counting experiments.

Thus, in § 4 we evaluate the clipped correlation function. Our result is not expressible as a simple analytic expression; nevertheless the approach yields an approximation good to any degree of approximation desired. We illustrate this technique by examining the clipped correlation function for the Brillouin spectrum in an approximation one order better than currently available results.

2. Formulation of the problem and formal solutions

The joint counting statistics describing the emitted photoelectrons from a detector exposed to an optical field can be characterized by the probability of counting n photoelectrons during a time interval T_1 centred at time t_1 and subsequently counting m electrons during time interval T_2 centred at t_2 which we write as $P(n_1, m_2)$. However an important quantity, the correlation function, is more useful from an experimentalist's point of view. The correlation function is defined as

$$\langle n_1 m_2 \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nmP(n_1, m_2). \tag{2.1}$$

An analysis of the photoemission process (Glauber 1968) reveals that the probability of counting n photoelectrons in a time interval T is given by

$$P(n, t) = \int_0^{\infty} \frac{1}{n!} \Omega^n \, \mathrm{e}^{-\Omega} P(\{\alpha_k\}) \prod_k \mathrm{d}^2 \alpha_k \tag{2.2}$$

where

$$E^{(-)}(r, t) = \sum_k (\tfrac{1}{2} h\omega_k)^{1/2} \alpha_k U_k(r) \exp(\mathrm{i}\omega_k t) \tag{2.3}$$

is the negative frequency part of the complex field expanded over the orthonormal mode functions $U_k(r)$, and

$$\Omega \equiv s \int_A \mathrm{d}^3 r' \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} \mathrm{d}t' E^{(-)}(r', t') E^{(+)}(r', t'), \tag{2.4}$$

where A is the area of the detector and s is the sensitivity of the detector (which is assumed to be independent of frequency). Denoting the average with respect to the weight function P by $\langle \ldots \rangle$, we can rewrite equation (2.2) in the more compact form

$$P(n, t) = \left\langle \frac{1}{n!} \Omega^n \, \mathrm{e}^{-\Omega} \right\rangle. \tag{2.5}$$

We can extend these results to joint probabilities. By an analysis similar to the one used to derive equation (2.2) (Jakeman 1970, Bédard 1967), it can be shown that

$$P(n_1, m_2) = \left\langle \frac{1}{n!}\Omega_1^n \, e^{-\Omega_1} \frac{1}{m!}\Omega_2^m \, e^{-\Omega_2} \right\rangle, \tag{2.6}$$

where

$$\Omega_i = s \int_A d^3r' \int_{t_i - \frac{1}{2}T}^{t_i + \frac{1}{2}T} dt' E^{(-)}(r', t') E^{(+)}(r', t'). \tag{2.7}$$

We will find it convenient to define a two-fold generating function $Q(\lambda_1, \lambda_2)$ by

$$Q(\lambda_1, \lambda_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (1-\lambda_1)^n (1-\lambda_2)^m P(n_1, m_2). \tag{2.8}$$

Provided that Q is known, then $P(n_1, m_2)$ can be evaluated by differentiation:

$$P(n_1, m_2) = \frac{(-1)^{n+m}}{n!m!} \frac{\partial^n}{\partial \lambda_1^n} \frac{\partial^m}{\partial \lambda_2^m} Q(\lambda_1, \lambda_2) \bigg|_{\lambda_1 = \lambda_2 = 1} \tag{2.9}$$

with the factorial moments of order (jl) given by

$$\langle n(n-1)\ldots(n-j+1)m(m-1)\ldots(m-l+1)\rangle = (-1)^{j+l} \frac{\partial^j}{\partial \lambda_1^j} \frac{\partial^l}{\partial \lambda_2^l} Q(\lambda_1, \lambda_2) \bigg|_{\lambda_1 = \lambda_2 = 0}. \tag{2.10}$$

Furthermore, the correlation function is simply

$$\langle n_1 m_2 \rangle = \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} Q(\lambda_1, \lambda_2) \bigg|_{\lambda_1 = \lambda_2 = 0}. \tag{2.11}$$

$Q(\lambda_1, \lambda_2)$ can be formally evaluated as a function of the field parameters; by applying equations (2.5) and (2.7) we have

$$Q(\lambda_1, \lambda_2) = \langle \exp(-\lambda_1 \Omega_1 - \lambda_2 \Omega_2) \rangle. \tag{2.12}$$

Assuming the field to be chaotic, then the P function is given by the gaussian

$$P(\{\alpha_k\}) = \prod_k \frac{1}{\pi \langle n_k \rangle} \exp\left(- \frac{|\alpha_k|^2}{\langle n_k \rangle} \right) \tag{2.13}$$

where $\langle n_k \rangle$ is the average occupation number of the kth mode.

Thus for a detector small in size compared to a coherence area, we can write

$$Q(\lambda_1, \lambda_2) = \int \prod_k \frac{d^2 \gamma_k}{\pi} \exp\{ -\boldsymbol{\gamma}^+ (\mathbf{I} + \mathbf{M}) \boldsymbol{\gamma} \} \tag{2.14}$$

where

$$M_{kk'} = \langle n_k \rangle^{1/2} \langle n_{k'} \rangle^{1/2} \tfrac{1}{2} h A s (\omega_k \omega_{k'})^{1/2} \times \left(\lambda_1 \int \exp\{ i(\omega_k - \omega_{k'})t \} \, dt + \lambda_2 \int \exp\{ i(\omega_k - \omega_{k'})t \} \, dt \right) U_k U_{k'} \tag{2.15}$$

and $\gamma_k \equiv \alpha_k / \langle n_k \rangle^{1/2}$. Since $\mathbf{M}$ is an (infinite) hermitian matrix it can be diagonalized by a unitary transformation

$$\mathbf{U}^+ \mathbf{M} \mathbf{U} = \boldsymbol{\Lambda}, \qquad \mathbf{U} \boldsymbol{\gamma} = \boldsymbol{\beta}. \tag{2.16}$$

Upon performing the necessary change of variables, we have

$$Q(\lambda_1, \lambda_2) = \int \prod_k \frac{d^2\beta_k}{\pi} \exp\{-\boldsymbol{\beta}^+(\mathbf{I}+\boldsymbol{\Lambda})\boldsymbol{\beta}\}$$

$$= \int \prod_k \frac{d^2\beta_k}{\pi} \exp\{-\beta_k^*(1+m_k)\beta_k\}$$

$$= \prod_{k=0}^{\infty} \frac{1}{(1+m_k)} = \{\det(\mathbf{I}+\mathbf{M})\}^{-1} \tag{2.17}$$

where $m_k = m_k(\lambda_1, \lambda_2)$ are the eigenvalues of $\mathbf{M}$.

Consequently the problem now becomes the evaluation of these eigenvalues. There are two possible approaches. The first is to truncate $\mathbf{M}$ to finite size and evaluate the eigenvalues via a diagonalization routine. The second is to transform the infinite matrix eigenvalue problem into an equivalent eigenvalue problem involving an homogeneous Fredholm integral equation of the second kind having a symmetric kernel. Actually the methods complement each other in that the matrix involves the lineshape directly whereas the integral equation involves the Fourier transform of the lineshape, the correlation function, as its kernel. Although we originally carried out some of the calculations by the matrix method the vast majority were performed in the context of the integral equation method and it is this procedure that we outline below.

The matrix eigenvalue problem can be shown to be equivalent to the following integral equation (since Jakeman has already performed the analysis we refer to his paper for details of the derivation):

$$\lambda_1 \int_{t_1-\frac{1}{2}T}^{t_1+\frac{1}{2}T} g(t'-t)\phi_j(t')\,dt' + \lambda_2 \int_{t_2-\frac{1}{2}T}^{t_2+\frac{1}{2}T} g(t'-t)\phi_j(t')\,dt' = \frac{m_j}{w}\phi_j(t), \tag{2.18}$$

where (in our notation) $g(t'-t)$ is the normalized correlation function of the field and where $w \equiv As\langle E^{(-)}E^{(+)}\rangle$ is the average counting rate of the detector. We seek only the eigenvalues m_j, the eigenfunctions are superfluous. This integral equation is not of the standard symmetric Fredholm type because of the presence of two domains of definition: $t_1-\frac{1}{2}T \leqslant t \leqslant t_1+\frac{1}{2}T$ and $t_2-\frac{1}{2}T < t < t_2+\frac{1}{2}T$. It is only when $t_1 = t_2$ that the two domains coalesce and we retrieve the standard type again. We now outline a numerical technique for solving the 'gap' integral equation which we have developed. It is probably easier to understand the rationale behind the method by first considering the short-time counting approximation.

Before doing so, it is convenient at this point to rewrite equation (2.18) in terms of dimensionless variables. The spectrum of the light will have some characteristic width γ which will be defined more precisely for particular spectra in the next section. We take as our dimensionless unit of time $\tau \equiv \gamma t$. In order to avoid a proliferation of symbols, we will still use T for the counting time but assume it measured in units of γ. The corresponding dimensionless variable for the average count rate is $v \equiv w/\gamma$.

Assume that T is small compared with the time over which $g(t)$ decays to its e folding value. Then we can approximate equation (2.18) by its mean value

$$\lambda_1 T g(t)\phi(0) + \lambda_2 T g(t-\tau)\phi(\tau) = \frac{m}{v}\phi(t). \tag{2.19}$$

This equation must be satisfied for $t = 0$, τ simultaneously. These two conditions are expressed in the form

$$\begin{pmatrix} \lambda_1 Tg(0) & \lambda_2 Tg(\tau) \\ \lambda_1 Tg(\tau) & \lambda_2 Tg(0) \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(\tau) \end{pmatrix} = \frac{m}{v} \begin{pmatrix} \phi(0) \\ \phi(\tau) \end{pmatrix} \tag{2.20}$$

where we have utilized the fact that $g(\tau) = g(-\tau)$. Then since $g(0) = 1$,

$$\prod_k (1+m_k) = \det \begin{pmatrix} 1+\lambda_1 Tv & \lambda_2 Tvg(\tau) \\ \lambda_1 Tvg(\tau) & 1+\lambda_1 Tv \end{pmatrix}. \tag{2.21}$$

Upon evaluating the determinant and noting that the left-hand side is the reciprocal of the double generating function, we have

$$Q(\lambda_1, \lambda_2) = \{1+(\lambda_1+\lambda_2)vT+\lambda_1\lambda_2(vT)^2(1+g^2(\tau))\}^{-1}. \tag{2.22}$$

This is the same expression that Bédard (1967) obtained for very short counting times.

To this degree of approximation, the correlation function is given by

$$\langle n_1 m_2 \rangle = \frac{\partial^2}{\partial\lambda_1 \partial\lambda_2} Q(\lambda_1\lambda_2)\bigg|_{\lambda=\lambda_2=0} = (vT)^2(1+g^2(\tau)). \tag{2.23}$$

We now generalize this approach by using N point Gauss quadrature to improve the approximation to the integrals in equation (2.18). This generalization, valid for arbitrarily long counting times, can be expressed in a form identical to equation (2.18) except that each element of the 2×2 matrix will itself be an $N \times N$ matrix. Using N point Gauss quadrature, we can approximate an integral by

$$\int_{-1}^{+1} f(x)\,\mathrm{d}x = \sum_{n=1}^{N} H_n f(x_n) \tag{2.24}$$

where H_n are the Gauss weight factors and x_n are the Gauss quadrature points. As is well known, N point Gauss quadrature is exact for any polynomial $f(x)$ of degree $(2N-1)$ or less (Kopal 1961).

Applying Gauss quadrature to the integral equation, equation (2.18) yields

$$\lambda_1 \sum_{i=1}^{N} H_i g(t-\tfrac{1}{2}Tx_i)\phi(\tfrac{1}{2}Tx_i)+\lambda_2 \sum_{i=1}^{N} H_i g(t-\tfrac{1}{2}Tx_i-\tau)\phi(\tfrac{1}{2}Tx_i+\tau) = \frac{2}{vT}\phi(t) \tag{2.25}$$

where we have transformed the variable limits of integration to the standard interval $(-1, +1)$. We require that this single equation be satisfied at the $2N$ values of t given by

$$t_j = \tfrac{1}{2}Tx_1, \ldots, \tfrac{1}{2}Tx_N, \tfrac{1}{2}Tx_1+\tau, \ldots, \tfrac{1}{2}Tx_N+\tau. \tag{2.26}$$

This yields $2N$ equations for the $2N$ unknowns $\phi(t_j)$, which, since the equations are homogeneous, requires the determination of the eigenvalues m_k. This system of equations can be cast into matrix form by defining the $N \times N$ matrix $\mathbf{B}_N(\tau)$:

$$\mathbf{B}_N(\tau) = \begin{pmatrix} h_{11}g(\tau) & h_{12}g\{\tfrac{1}{2}T(x_2-x_1)+\tau\} & \ldots & h_{1N}g\{\tfrac{1}{2}T(x_N-x_1)+\tau\} \\ h_{21}g\{\tfrac{1}{2}T(x_1-x_2)+\tau\} & h_{22}g(\tau) & \ldots & h_{2N}g\{\tfrac{1}{2}T(x_N-x_2)+\tau\} \\ \vdots & \vdots & \ddots & \\ h_{N1}g\{\tfrac{1}{2}T(x_1-x_N)+\tau\} & h_{N2}g\{\tfrac{1}{2}T(x_2-x_N)+\tau\} & \ldots & h_{NN}g(\tau) \end{pmatrix} \tag{2.27}$$

where $h_{ij} \equiv (H_i H_j)^{1/2}$, and the column matrix $\boldsymbol{\phi}_N(\tau)$ is given by

$$\boldsymbol{\phi}_N(\tau) = \begin{pmatrix} \phi(\frac{1}{2}Tx_1 + \tau) \\ \phi(\frac{1}{2}Tx_2 + \tau) \\ \vdots \\ \phi(\frac{1}{2}Tx_N + \tau) \end{pmatrix}. \tag{2.28}$$

Consequently the requirement that equation (2.25) holds for the $2N$ values of t given by equation (2.26) can be succinctly written as

$$\begin{pmatrix} \mathbf{I}_N + \frac{1}{2}\lambda_1 vT\mathbf{B}_N(0) & \frac{1}{2}\lambda_2 vT\mathbf{B}_N(-\tau) \\ \frac{1}{2}\lambda_1 vT\mathbf{B}_N(\tau) & \mathbf{I}_N + \frac{1}{2}\lambda_2 vT\mathbf{B}_N(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_N(0) \\ \boldsymbol{\phi}_N(\tau) \end{pmatrix} = 0 \tag{2.29}$$

where $\mathbf{I}_N$ is the unit matrix of order N. However, the determinant of the coefficient matrix is simply the double generating function $Q(\lambda_1\lambda_2)$

$$\begin{aligned} Q(\lambda_1\lambda_2)^{-1} &= \prod_k (1+m_k) \\ &= \det\{\mathbf{I}_N + \tfrac{1}{2}vT(\lambda_1+\lambda_2)\mathbf{B}_N(0) + (\tfrac{1}{2}vT)^2\lambda_1\lambda_2(\mathbf{B}_N(0)\mathbf{B}_N(0) - \mathbf{B}_N(\tau)\mathbf{B}_N(-\tau))\}. \end{aligned} \tag{2.30}$$

From equation (2.30) one can, in principle, extract all the joint probabilities by differentiation. We have evaluated some of the lower order joint densities and in particular $P(0, 0) = Q(1, 1)$ for several spectra (work to be published). $P(0, 0)$ is of particular interest as it can be measured experimentally over a wide variety of counting times (Furcinitti *et al* 1972).

The second-order quantity of most interest is the correlation function (see equation (2.11)). We can perform the indicated differentiation of $Q(\lambda_1\lambda_2)$ to evaluate the correlation function, the details are given in the appendix. The final result is

$$\langle n_1 m_2 \rangle = (\tfrac{1}{2}vT)^2\{(\operatorname{tr}\mathbf{B}_N(0))^2 + \operatorname{tr}(\mathbf{B}_N(\tau)\mathbf{B}_N(-\tau))\} \tag{2.31}$$

which reduces to equation (2.23) in the case where $N = 1$.

This expression can be further simplified by noting that

$$\operatorname{tr}\mathbf{B}(0) = \sum_j B_{jj}(0) = 2. \tag{2.32}$$

Accordingly, equation (2.31) becomes

$$\langle n_1 m_2 \rangle = (vT)^2(1 + \tfrac{1}{4}\operatorname{tr}\mathbf{B}(\tau)\mathbf{B}(-\tau)). \tag{2.33}$$

This form of the background term is required by the fact that

$$\lim_{\tau\to\infty} \langle n_1 m_2 \rangle = \langle n \rangle \langle n \rangle = (vT)^2. \tag{2.34}$$

Equation (2.33) can be cast into a more familiar form by examining the limit as N approaches infinity. Since

$$\begin{aligned} \lim_{N\to\infty} \operatorname{tr}(\mathbf{B}(\tau)\mathbf{B}(-\tau)) &= \lim_{N\to\infty} \sum_j^N \sum_k^N H_j H_k \{g(\tfrac{1}{2}(x_k - x_j)T + \tau)\}^2 \\ &= \frac{4}{T^2}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt'\, dt''(g(t'-t''+\tau))^2 \end{aligned} \tag{2.35}$$

we see that

$$\langle n_1 m_2 \rangle = (vT)^2 \left(1 + \frac{1}{T^2} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} (g(t'-t''+\tau))^2 \, dt' \, dt'' \right). \tag{2.36}$$

In accordance with standard practice, we normalize the photoelectron correlation function by dividing by $(vT)^2 = \langle n \rangle^2$:

$$c(\tau) = \frac{\langle n(t) n(t+\tau) \rangle}{\langle n(t) \rangle^2}. \tag{2.37}$$

We see then that

$$c(\tau) = \frac{1}{T^2} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} (g(t'-t''+\tau))^2 \, dt' \, dt'' + 1. \tag{2.38}$$

Note that in the short-counting-time approximation ($N = 1$)

$$\lim_{T \to 0} c(\tau) = |g(\tau)|^2 + 1. \tag{2.39}$$

The correlation function expressed in equation (2.38) is equivalent to the result given in equation (6.13) of Mehta's review article. Our derivation is entirely quantum mechanical *à la* Glauber; whereas the result in Mehta is based upon semiclassical theory.

3. Photoelectron correlation function for Brillouin spectrum

Before we discuss the clipped photoelectron correlation function it is of some interest to present numerical results for an actual spectrum. While current experimental uses of photoelectron counting statistics measurements have generally been limited to measuring the bandwidths of single (usually Lorentz) spectral lines, physical phenomena giving rise to several spectral lines are also of interest. In the electrophoresis experiments of Ware and Flygare (1971), for example, a Lorentz line of different frequency is produced for each component of distinct electrophoretic mobility. Another example is Brillouin scattering from media where the velocity of sound is very small such as in the work of Katyl and Ingard (1968) on surface waves along a liquid interface.

We consider a spectrum consisting of the superposition of two Lorentz lines symmetrically placed about a central Lorentz line of different height and halfwidth. This spectrum may be written in the form:

$$I(\omega) = \sum_{j=-1}^{+1} R_j \left(\frac{\gamma_j / \pi}{(\omega - \omega_j)^2 + \gamma_j^2} \right). \tag{3.1}$$

The total energy in the beam is normalized to unity:

$$\int_{-\infty}^{\infty} I(\omega) \, d\omega = \sum_{j=-1}^{+1} R_j = 1. \tag{3.2}$$

Since $R_1 = R_{-1}$, we define a parameter $\alpha = 2R_1$ which represents the fraction of the total energy contained in the Brillouin lines. Thus $\alpha = 0$ describes a single (Rayleigh) line and $\alpha = 1$ describes a pair of (Brillouin) lines in the absence of a Rayleigh line. The actual Brillouin lines are spaced symmetrically about the Rayleigh line so that $\omega_1 - \omega_0 = \omega_0 - \omega_{-1}$. We also define a dimensionless quantity $\Delta \equiv (\omega_1 - \omega_0)/\gamma_0$.

Similarly, we define $\delta \equiv \gamma_1/\gamma_0 = \gamma_{-1}/\gamma_0$ which gives the ratio of the width of the Brillouin line to the Rayleigh line. We will measure time in units of $1/\gamma_0$, setting $t = \gamma_0 t$, and measure beam strength in units of γ_0 by setting $v = w/\gamma_0$. A summary of these parameters is given in table 1.

Table 1. Summary of the dimensionless parameters

Symbol	Definition	Meaning
α	$2R_{-1} = 2R_1 = 1-R_0$	fraction of energy in Brillouin lines
Δ	$(\omega_1 - \omega_0)/\gamma_0$	ratio of frequency shift to Rayleigh halfwidth
δ	$\gamma_1/\gamma_0 = \gamma_{-1}/\gamma_0$	ratio of Brillouin to Rayleigh halfwidths
v	w/γ_0	number of photocounts in one correlation time γ_0^{-1}
τ	$t\gamma_0$	counting time measured in units of correlation time

The correlation function of the Brillouin spectrum is

$$g(\tau) = \exp(-i\omega_0 t)\{(1-\alpha)\,e^{-\tau} + \alpha\, e^{-\delta\tau}\cos\Delta\tau\}. \tag{3.3}$$

We can drop the factor $\exp(-i\omega_0 t)$ in accordance with the analysis of Barakat and Glauber (1973) who have shown that the central frequency ω_0 does not enter into the analysis when the spectrum is quasimonochromatic.

A superposition of spectral lines, such as the Brillouin spectrum, will display hetrodyning thus causing $c(\tau)$ to oscillate on a time scale of $1/\Delta$. $c(\tau)$ is plotted as a function of τ for counting times of 0·1 and 0·2 in figures 1 and 2 respectively. We selected these

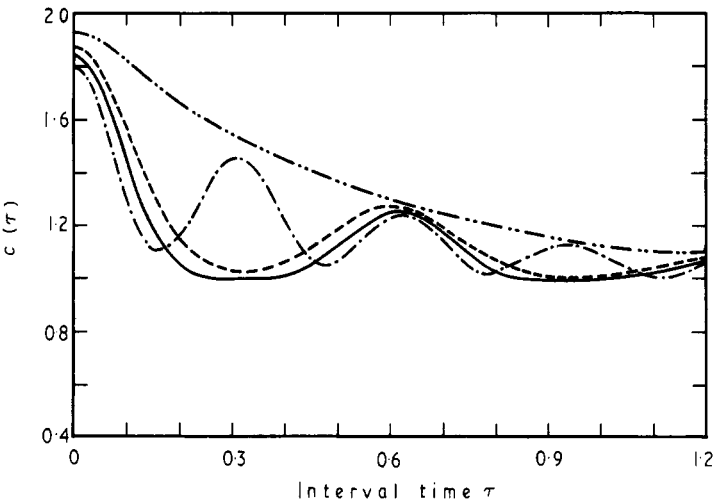

Figure 1. Photoelectron correlation function $c(\tau)$ for a Brillouin spectrum ($\Delta = 10$, $\delta = 1$) with $T = 0{\cdot}1$, $v = 1$: —··— $\alpha = 0$, – – – – $\alpha = 0{\cdot}4$, — $\alpha = 0{\cdot}6$, —·— $\alpha = 1{\cdot}0$.

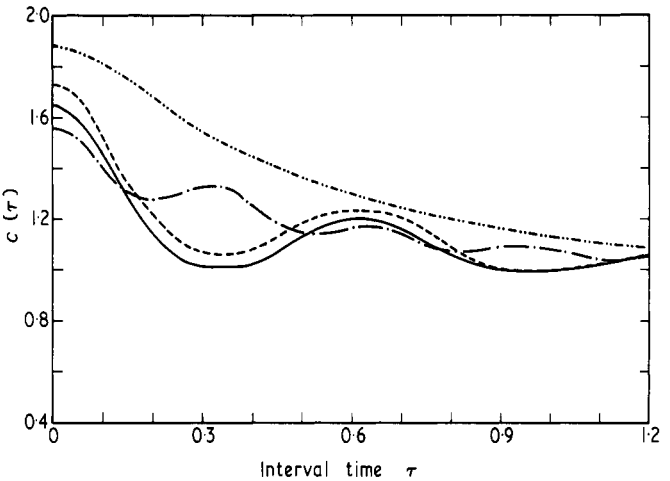

Figure 2. Photoelectron correlation function $c(\tau)$ for a Brillouin spectrum ($\Delta = 10$, $\delta = 1$) with $T = 0{\cdot}2$, $v = 1$: —··— $\alpha = 0$, - - - - $\alpha = 0{\cdot}4$, — $\alpha = 0{\cdot}6$, —·— $\alpha = 1{\cdot}0$.

counting times because 0·1 is long enough to show departure from the short-counting-time approximation, and 0·2 is just long enough to cause averaging over major details of the spectrum. For counting times much longer than $1/\Delta$, $c(\tau)$ tends to a gaussian-like shape and the oscillations vanish altogether.

When the spectral line is lorentzian so that the correlation function is

$$g(\tau) = e^{-|\tau|} \tag{3.7}$$

the integral can be evaluated directly yielding

$$c(\tau) = \begin{cases} 1 + e^{-2\tau}\left(\dfrac{\sinh T}{T}\right)^2 & 0 \leqslant T \leqslant \tau \\ 1 + T^{-2}\left(\frac{1}{4}e^{-2(T+\tau)} + \frac{1}{4}e^{-2(T-\tau)} - \frac{1}{2}e^{-2T} + T - \tau\right) & 0 \leqslant \tau \leqslant T. \end{cases} \tag{3.8}$$

This equation can be used to confirm that the accuracy of equation (2.33) is better than 0·1 % for the case of the Lorentz spectrum. The integral can, of course, be explicitly evaluated for the Brillouin spectrum, but equation (2.33) is easier to apply in practice.

4. Clipped correlation function

Jakeman and Pike (1969) have pointed out that one can apply the method of 'clipping' a fluctuating signal in performing a photoelectron correlation experiment, and that this would permit a considerable simplification in the electronic equipment. Defining the clipped correlation function by

$$C_k(\tau) \equiv \langle n_k(0) m(\tau) \rangle \tag{4.1}$$

where

$$n_k(\tau) = \begin{cases} 0 & \text{if} \quad n(\tau) < k \\ 1 & \text{if} \quad n(\tau) \geqslant k \end{cases} \tag{4.2}$$

we see that measuring $C_k(\tau)$ requires multiplication by only ones and zeros.

The clipped correlation function has been examined in the short-counting approximation ($N = 1$ in the notation of § 2) by Jakeman *et al* (1970), Degiorgio and Lastovka (1971), Kelly (1971); in the case of a Lorentz spectrum for arbitrary counting time by Jakeman (1970); and in several specialized cases by Koppel (1971). In order to demonstrate the use of equation (2.30) for the generating function we shall evaluate the clipped correlation function in an approximation one order better than $N = 1$ in the case of a Brillouin spectrum. This means that we take $N = 2$ in the generating function.

To obtain $C_k(\tau)$ from the double generating function, we employ equations (2.9) and (2.10):

$$C_k(\tau) = \sum_{m=0}^{\infty}\sum_{n=k}^{\infty} mP(n,m) = -\frac{\partial}{\partial\lambda_2}\sum_{n=k}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}Q(\lambda_1,\lambda_2)\bigg|_{\lambda_1=1,\lambda_2=0}. \tag{4.3}$$

For $N = 1$, we recall that the generating function is given by equation (2.23). In this case, the differentiation and subsequent summation can be carried out directly by using the substitutions

$$\begin{aligned} U &= 1+vT\lambda_2 \\ V &= vT+(vT)^2(1-g^2(\tau))\lambda_2 . \end{aligned} \tag{4.4}$$

With Q now expressible as $(U+\lambda_1 V)^{-1}$, we have

$$\frac{\partial^n}{\partial\lambda_1^n}Q(\lambda_1,\lambda_2)\bigg|_{\lambda_1=1} = n!(-1)^n\frac{(V)^n}{(U+V)^{n+1}} \tag{4.5}$$

and the sum over n can be evaluated easily:

$$\sum_{n=k}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}Q\bigg|_{\lambda_1=1} = \frac{1}{U}\left(\frac{V}{U+V}\right)^k. \tag{4.6}$$

Upon performing the remaining differentiation, we have

$$C_k(\tau) = \left(\frac{vT}{1+vT}\right)^k vT\left(1+\frac{k}{1+vT}g^2(\tau)\right). \tag{4.7}$$

In order to obtain an expression for $C_k(\tau)$ valid for longer counting times, we use the generating function given in equation (2.30) with $N = 2$. In this case we can show that

$$\begin{aligned} &Q(\lambda_1,\lambda_2)^{-1} \\ &\quad = \det\begin{pmatrix} 1+s_1+s_2+s_1s_2(1+Y^2-X^2-W^2) & (s_1+s_2)Y+s_1s_2(2Y-XU-XW) \\ (s_1+s_2)Y+s_1s_2(2Y-XU-XW) & 1+s_1+s_2+s_1s_2(1+Y^2-X^2-U^2) \end{pmatrix} \end{aligned} \tag{4.8}$$

where $s_j \equiv (\frac{1}{2}vT)\lambda_j$ and

$$\begin{aligned} X &= g(\tau), & U &= g(\tau+\Delta) \\ Y &= g(\Delta), & W &= g(\tau-\Delta) \end{aligned} \tag{4.9}$$

with $\Delta \equiv \frac{1}{2}(x_2-x_1)T$. We can evaluate the determinant explicitly with the result that

$$Q = (L+Ms_2+\mathrm{O}(s_2^2))^{-1} \tag{4.10}$$

where

$$L = 1+2s_1+(1-Y^2)s_1^2 \tag{4.11}$$

$$M = 2+s_1(4-2X^2-U^2-W^2)+s_1^2(2-2Y^2-2X^2-U^2-W^2+2XYU+2XYW). \tag{4.12}$$

Thus

$$\left.\frac{\partial Q}{\partial\lambda_2}\right|_{\lambda_2=0} = -\left(\frac{vT}{2}\right)\frac{M}{L^2}. \tag{4.13}$$

We must now evaluate the sum over n:

$$C_k(\tau) = -\left(\frac{vT}{2}\right)\sum_{n=k}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}\left(-\frac{M}{L^2}\right)\Bigg|_{\lambda_1=1} \tag{4.14}$$

Unfortunately we have not found any way to carry out this summation in closed form. For small k we can use a trick due to Jakeman (1970, appendix 3). Since

$$-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}\frac{\partial}{\partial\lambda_2}Q(\lambda_1\lambda_2)\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}mP(n,m) = \langle n\rangle, \tag{4.15}$$

we can rewrite the clipped correlation function as

$$\begin{aligned} C_k(\tau) &= -\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}\frac{\partial}{\partial\lambda_2}Q(\lambda_1\lambda_2)\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} + \sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}\frac{\partial}{\partial\lambda_2}Q(\lambda_1\lambda_2)\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} \\ &= \langle n\rangle+\frac{vT}{2}\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda_1^n}\left(-\frac{M}{L^2}\right)\Bigg|_{\lambda_1=1}. \end{aligned} \tag{4.16}$$

Equation (4.16) is easy to apply for small k; in fact

$$\begin{aligned} C_1(\tau) &= \langle n\rangle+\left(\frac{vT}{2}\right)\left(-\frac{M}{L^2}\right) \\ C_2(\tau) &= C_1(\tau)-\left(\frac{vT}{2}\right)^2\left(-\frac{M'}{L^2}+\frac{2ML'}{L^3}\right) \\ C_3(\tau) &= C_2(\tau)+\frac{1}{2}\left(\frac{vT}{2}\right)^3\left(-\frac{M''}{L^2}+\frac{4M'L'+2ML''}{L^3}-\frac{6ML'^2}{L^4}\right) \\ C_4(\tau) &= C_3(\tau)-\frac{1}{3!}\left(\frac{vT}{2}\right)^4\left(\frac{6L''M'-6M''L'}{L^3}-\frac{18M'L'^2+18MLL''}{L^4}+\frac{24L'^3M}{L^5}\right) \end{aligned} \tag{4.17}$$

where the prime denotes differentiation with respect to s and all expressions are to be evaluated at $s = \frac{1}{2}vT$. $C_k(\tau)$ satisfies the recurrence relation

$$C_k(\tau) = C_{k-1}(\tau)+\frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda_1^{k-1}}\frac{\partial}{\partial\lambda_2}Q(\lambda_1\lambda_2)\Bigg|_{\substack{\lambda_1=1\\ \lambda_2=0}} \qquad k>1. \tag{4.18}$$

Evaluating $C_k(\tau)$ in this manner introduces an error dependent upon the counting time T because we have taken $N = 2$ in the evaluation of the generating function. We estimated the severity of this error by comparing calculations of $c(\tau)$ for $N = 2$ with those made using large values of N for several different spectra. In the worst situation

(Brillouin spectrum with $\Delta = 10$) we found that $N = 2$ gave an accuracy of better than 3% for $T = 0{\cdot}2$ and 0·5% for $T = 0{\cdot}1$. We would not expect the error introduced into $C_k(\tau)$ to be very much different. The situation is much better, incidentally, for single spectral lines.

The clipped correlation function $c_1(\tau)$ for a Brillouin spectrum is shown in figures 3 and 4. As is customary (Jakeman and Pike 1969) we have normalized by dividing by the product of the true and clipped mean counting rates:

$$c_k(\tau) \equiv \frac{C_k(\tau)}{\langle n_k \rangle \langle n \rangle}. \tag{4.19}$$

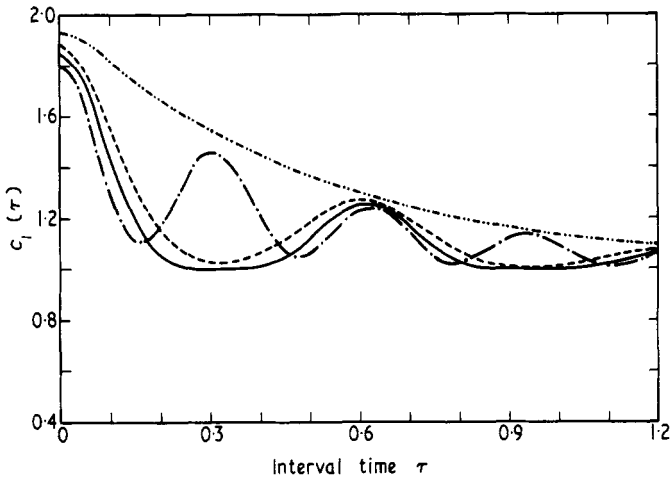

Figure 3. Clipped photoelectron correlation function $c_1(\tau)$ for a Brillouin spectrum ($\Delta = 10$, $\delta = 1$) with $T = 0{\cdot}1$, $v = 2{\cdot}0$ ($vT = 0{\cdot}2$): —··— $\alpha = 0$, - - - $\alpha = 0{\cdot}4$, — $\alpha = 0{\cdot}6$, —·— $\alpha = 1{\cdot}0$.

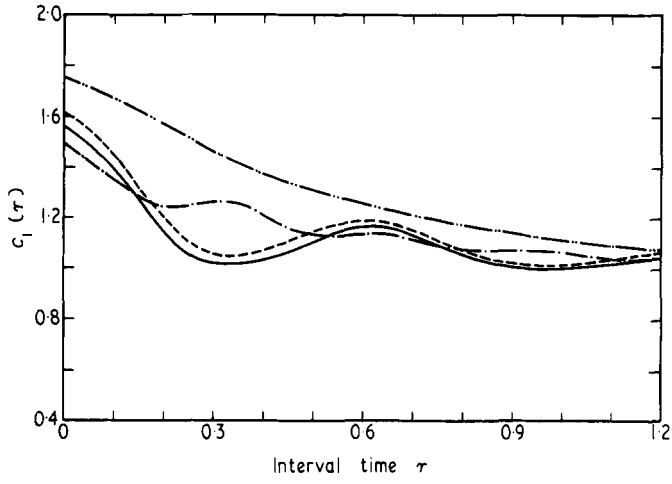

Figure 4. Clipped photoelectron correlation function $c_1(\tau)$ for a Brillouin spectrum ($\Delta = 10$, $\delta = 1$) with $T = 0{\cdot}2$, $v = 1{\cdot}0$ ($vT = 0{\cdot}2$): —··— $\alpha = 0$, - - - $\alpha = 0{\cdot}4$, — $\alpha = 0{\cdot}6$, —·— $\alpha = 1{\cdot}0$.

We note that as T becomes larger there is a decrease in the slope of $c_1(\tau)$ at the origin, and an averaging over details in the spectrum. $T = 0{\cdot}2$ is large enough to separate the curves of different α values from each other at $\tau = 0$.

In order to give some idea of the effects of clipping at different levels we plotted in figure 5 $c_k(\tau)$ for a Lorentz spectrum at $T = 0{\cdot}1$.

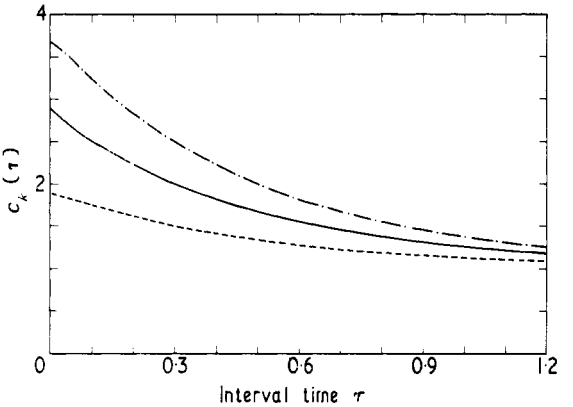

Figure 5. $c_k(\tau)$ for a Lorentz spectrum with $T = 0{\cdot}1$, $v = 0{\cdot}1$: --- $k = 1$, — $k = 2$, —·— $k = 3$.

The method employed in this section can be directly applied to any other quasi-monochromatic spectrum, of course. By using larger values of N one can obtain any accuracy desired (or alternatively, treat longer counting times), though for $k > 2$ the algebra becomes considerably more cumbersome.

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Appendix

Let us outline the derivation of the correlation function (equation (2.11)). From equations (2.17) and (2.30), we have

$$\langle n_1 m_2 \rangle = \frac{\partial^2}{\partial\lambda_1 \partial\lambda_2} (\det \mathbf{A})^{-1} \tag{A.1}$$

where

$$\mathbf{A} \equiv \mathbf{I} + \tfrac{1}{2}vT(\lambda_1 + \lambda_2)\mathbf{B}(0) + (\tfrac{1}{2}vT)^2 \lambda_1 \lambda_2 (\mathbf{B}(0)\mathbf{B}(0) - \mathbf{B}(\tau)\mathbf{B}(-\tau)). \tag{A.2}$$

The matrix differentiation can be explicitly performed. By virtue of the chain rule (let $\det \mathbf{A} \equiv A$ for convenience):

$$\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}A^{-1} = \left[\frac{2}{A^3}\left(\frac{\partial}{\partial\lambda_1}A\right)\left(\frac{\partial}{\partial\lambda_2}A\right) - \frac{1}{A^2}\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}A\right]_{\lambda_1=\lambda_2=0} \tag{A.3}$$

and since

$$\mathbf{A}|_{\lambda_1=\lambda_2=0} = \mathbf{I} \tag{A.4}$$

we have the right-hand side of equation (A.3) given by

$$2\left(\frac{\partial}{\partial\lambda_1}A\right)\left(\frac{\partial}{\partial\lambda_2}A\right) - \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}A, \qquad (\lambda_1 = \lambda_2 = 0). \tag{A.5}$$

By writing

$$\det \mathbf{A} = \exp(\operatorname{tr} \ln \mathbf{A}) \tag{A.6}$$

the following two formulae can be derived:

$$\frac{\partial}{\partial\lambda_1}A = A \operatorname{tr} \mathbf{A}^{-1}\frac{\partial}{\partial\lambda_1}\mathbf{A}, \tag{A.7}$$

$$\begin{aligned}\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}A = A\Bigg\{&(\operatorname{tr}\mathbf{A}^{-1})\frac{\partial}{\partial\lambda_1}\mathbf{A}(\operatorname{tr}\mathbf{A}^{-1})\frac{\partial}{\partial\lambda_2}\mathbf{A}\\ &+(\operatorname{tr}\mathbf{A}^{-1})\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\mathbf{A} - (\operatorname{tr}\mathbf{A}^{-2})\left(\frac{\partial}{\partial\lambda_1}\mathbf{A}\right)\left(\frac{\partial}{\partial\lambda_2}\mathbf{A}\right)\Bigg\}\end{aligned} \tag{A.8}$$

When specialized to equation (A.4), they read

$$\frac{\partial}{\partial\lambda_1}A = \tfrac{1}{2}vT\mathbf{B}(0) + (\tfrac{1}{2}vT)^2\lambda_2(\mathbf{B}(0)\mathbf{B}(0) - \mathbf{B}(\tau)\mathbf{B}(-\tau)) \tag{A.9}$$

$$\frac{\partial}{\partial\lambda_2}A = \tfrac{1}{2}vT\mathbf{B}(0) + (\tfrac{1}{2}vT)^2\lambda_1(\mathbf{B}(0)\mathbf{B}(0) - \mathbf{B}(\tau)\mathbf{B}(-\tau)) \tag{A.10}$$

$$\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}A = (\tfrac{1}{2}vT)^2(\mathbf{B}(0)\mathbf{B}(0) - \mathbf{B}(\tau)\mathbf{B}(-\tau)). \tag{A.11}$$

When these expressions are substituted into equation (A.5) we find

$$\langle n_1 m_2\rangle = (\tfrac{1}{2}vT)^2\{(\operatorname{tr}\mathbf{B}(0))^2 - \operatorname{tr}\mathbf{B}(\tau)\mathbf{B}(-\tau)\} \tag{A.12}$$

which is equation (2.31).

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